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Multilinear Singular Value Decomposition for Two Qubits

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ABSTRACT

Schmidt decomposition has been used in the local unitary (LU) classification of bipartite quantum states for some time. In order to generalize the LU classification of bipartite quantum states into multipartite quantum states, higher order singular value decomposition (HOSVD) is introduced but specific examples have not been explicitly worked out. In this pedagogical paper, we would like to work out such details in two qubits since the LU classification of two qubits is well known. We first demonstrate the method of HOSVD in two-qubit systems and discuss its properties. In terms of the LU classification of two-qubit states, some subtle differences in the stabilizer groups of entanglement classes are noticed when Schmidt decomposition is substituted by HOSVD. To reconcile the differences between the two, further studies are needed.

Keywords: Higher order singular value decomposition, two qubits, local unitary operation.

1. Introduction

Classical computers process information using binaries which are deterministic, i.e. either 0 or 1 but not both at the same time. These binaries are called bits. In the early 1980s, the consideration of quantum mechanics on computation and information science sprouted the idea of the quantum bits or simply qubits, which is the analogus version of classical bits in quantum mechanical sense.

A qubit is a two-level quantum system with quantum state vector

$$|\varphi\rangle = \varphi_1 |1\rangle + \varphi_2 |2\rangle \tag{1}$$

such that $|1\rangle$ and $|2\rangle$ are the basis, φ_1 and φ_2 are the probability amplitude with

$$|\varphi_1|^2 + |\varphi_2|^2 = 1. \tag{2}$$

The basis vectors are defined in such a way to match the indices of the tensor elements. By convention, we also let

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3}$$

Therefore, equation (1) can also be written as

$$|\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{4}$$

It is not uncommon to manipulate quantum systems of more than one qubit in quantum computation. Two qubits can be combined together by tensor product operation such that the Hilbert space becomes the tensor product of one another, $\mathcal{H} \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. A general two-qubit state is given as

$$|\psi\rangle = \psi_{11} |11\rangle + \psi_{12} |12\rangle + \psi_{21} |21\rangle + \psi_{22} |22\rangle \tag{5}$$

with the basis vectors $|11\rangle$, $|12\rangle$, $|21\rangle$, $|22\rangle$, and

$$|\psi_{11}|^2 + |\psi_{12}|^2 + |\psi_{21}|^2 + |\psi_{22}|^2 = 1. \tag{6}$$

Tensor product in matrices is essentially Kronecker product. Therefore, for example

$$|12\rangle = |1\rangle \otimes |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

Two-qubit states can then be represented by

$$|\psi\rangle = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{21} \\ \psi_{22} \end{pmatrix}. \tag{8}$$

From the extension of this idea, we find that column matrices is one of the natural representations of the state vectors.

Unlike the classical counterpart, there exists non-local correlation between the quantum subsystems when two or more subsystems are taken into account. This correlation is called entanglement and it is considered as a resource in quantum computation. Hence, it is desirable to classify quantum states of a given quantum system according to operations that do not alter entanglement between the subsystems. Such kind of operations are necessarily local. Also, operations acting on the quantum systems must not change the probability amplitude of the quantum states, therefore it has to be unitary. We refer these kind of operations as local unitary (LU) operations.

In two-qubit LU classification scheme, it is typical to treat $\Psi = [\psi_{ij}]$ as matrices and apply matrix operations on it so that calculations can be simplified considerably (Carteret and Sudbery, 2000). Since square matrices will always have their Schmidt decomposition, Schmidt coefficients is also used to study the problem of separability between two qubits (Rudolph, 2005). With Schmidt decomposition, one can extend the LU classification from two qubits to $N \times N$ bipartite quantum states (Sinołęcka et al., 2002). However, when we move to multipartite quantum systems, the simplest being the three-qubit systems, $\Theta = [\vartheta_{ijk}]$ becomes a tensor (L. H. Lim. In L. Hogben, 2013). It is found that tensors cannot be decomposed by the same manner as in two-qubit case in general (Peres, 1995).

In 2000, de Lathauwer et al. formalized a generalized multilinear singular value decomposition, called higher order singular value decomposition (HOSVD), where it can be applied to tensors. Similar idea was reviewed, along with other tensor methods (Kolda and Bader, 2009). In particular, the method of HOSVD is adopted by Bin et al. (2012) and Jun-Li and Cong-Feng (2013) in their LU classification scheme to compare whether two multipartite pure quantum states are LU equivalent or not. In order to complement the existing literature regarding LU classification using HOSVD, we aim to provide a pedagogical approach on the topic using quantum system for which its LU classification results are widely known, i.e. two qubits pure states.

This paper is divided into three sections. Section 2 discusses about the mathematical tools needed in this research. In Section 3, main discoveries using HOSVD will be stated. The stabilizers for each of the two-qubit entanglement classes will also be stated and compared with Carteret and Sudbery's work. Conclusion will be made in Section 4.

Unless specified, tensors are written as calligraphic letters or capitalized Greek letters (Φ, Ψ, Θ, $\mathcal{T}$, $\mathcal{X}, \ldots$), while small italicized letters with subscripts will represent the tensor elements (φ_i, ψ_{ij}, ϑ_{ijk}, $t_{ij}, \ldots$). Meanwhile, italicized small letters are used to denote the varying indices (i, j, k, n, α, $\beta, \ldots$) and capital-italicized letters are used to indicate the index's upper bound (I, J, K, N, M_1, M_2, $M, \ldots$). Exceptions are found when writing the matrices, for example P, Q, R, X, Y, U, V, and $U^{(n)}$. In that case, a change in notation will be informed beforehand.

2. Theory

2.1 Vectors, matrices and tensors

Beside the column matrix representation in equations (4) and (8), we note that from one-qubit states (1) to two-qubit states (5), the probability amplitude of the quantum states changes from an one-index object to a two-index object. In general, one-index object $\Phi = [\varphi_i]$ is called tensor of order 1, while two-index object $\Psi = [\psi_{ij}]$ is called tensor of order 2.

Tensor of order 1 is usually represented as a column vector. In one-qubit case (1), it coincides with the state vector representation (4) written as

$$\Phi = [\varphi_i] = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{9}$$

Meanwhile, the representation of a tensor of order 2 is a matrix. In two-qubit case (5),

$$\Psi = [\psi_{ij}] = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}. \tag{10}$$

It is possible to include more qubits by tensor product, but now the probability amplitude of the multipartite composite quantum system will become a *higher order tensor*.

2.2 Higher order tensors and its matrix representations

As we increase the number of orders of a tensor, problem arises since we do not have a formal way to represent and analyze higher order tensors. However, based on the multiplication rules of tensor elements, de Lathauwer et al. in 2000 introduced a technique called *matrix unfolding* to represent higher order tensors by matrices.

Definition 1 (Matrix unfolding)**.** Assume an Nth-order complex tensor $\mathcal{X} \in \mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes \ldots \otimes \mathbb{C}^{I_N}$. The n-th matrix unfolding, $X_{(n)}$, which is the element in $\mathbb{C}^{I_n \times (I_{n+1} \times I_{n+2} \times \ldots \times I_N \times I_1 \times I_2 \times \ldots \times I_{n-1})}$, will contain the tensor element $\chi_{i_1 i_2 \ldots i_N}$ at the position with row number i_n and column number equal to (de Lathauwer et al., 2000)

$$\begin{aligned}&(i_{n+1}-1)I_{n+2}I_{n+3}\ldots I_N I_1 I_2 \ldots I_{n-1} + (i_{n+2}-1)I_{n+3}I_{n+4}\ldots I_N I_1 I_2 \ldots I_{n-1} \\ &+ \ldots + (i_N - 1)I_1 I_2 \ldots I_{n-1} + (i_1 - 1)I_2 I_3 \ldots I_{n-1} + (i_2 - 1)I_3 I_4 \ldots I_{n-1} \\ &+ \ldots + i_{n-1}. \end{aligned} \tag{11}$$

Note that the tensor product of vector spaces $\mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes \ldots \otimes \mathbb{C}^{I_N}$ is isomorphic to $\mathbb{C}^{I_1 \times I_2 \times \ldots \times I_N}$. In Definition 1, the notation $\mathbb{C}^{I_n \times (I_{n+1} \times \ldots \times I_N \times I_1 \times \ldots \times I_{n-1})}$ is used to show explicitly that for an n-th matrix unfolding, we will have an $I_n \times (I_{n+1} \times I_{n+2} \times \ldots \times I_N \times I_1 \times I_2 \times \ldots \times I_{n-1})$ matrix.

As an example, consider a tripartite quantum system with third-order tensor $\Theta = [\vartheta_{ijk}] \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ and quantum states written as

$$\begin{aligned}|\vartheta\rangle = {}& \vartheta_{111}|111\rangle + \vartheta_{112}|112\rangle + \vartheta_{113}|113\rangle + \vartheta_{121}|121\rangle + \vartheta_{122}|122\rangle + \vartheta_{123}|123\rangle \\ &+ \vartheta_{211}|211\rangle + \vartheta_{212}|212\rangle + \vartheta_{213}|213\rangle + \vartheta_{221}|221\rangle + \vartheta_{222}|222\rangle \\ &+ \vartheta_{223}|223\rangle, \end{aligned} \tag{12}$$

where $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3 \cong \mathbb{C}^{2\times 2\times 3} \equiv \mathbb{C}^{12}$.

The matrix unfoldings are as follow:-

- First matrix unfolding, $\Theta_{(1)}$
  Row index: i_1; Column index: $(i_2 - 1)I_3 + i_3$

$$\Theta_{(1)} \in \mathbb{C}^{2\times(2\times 3)};\ \Theta_{(1)} = \left(\begin{array}{ccc|ccc} \vartheta_{111} & \vartheta_{112} & \vartheta_{113} & \vartheta_{121} & \vartheta_{122} & \vartheta_{123} \\ \vartheta_{211} & \vartheta_{212} & \vartheta_{213} & \vartheta_{221} & \vartheta_{222} & \vartheta_{223} \end{array}\right)$$

- Second matrix unfolding, $\Theta_{(2)}$
  Row index: i_2; Column index: $(i_3-1)I_1+i_1$

$$\Theta_{(2)} \in \mathbb{C}^{2\times(3\times 2)};\ \Theta_{(2)} = \left(\begin{array}{cc|cc|cc} \vartheta_{111} & \vartheta_{211} & \vartheta_{112} & \vartheta_{212} & \vartheta_{113} & \vartheta_{213} \\ \vartheta_{121} & \vartheta_{221} & \vartheta_{122} & \vartheta_{222} & \vartheta_{123} & \vartheta_{223} \end{array}\right)$$

- Third matrix unfolding, $\Theta_{(3)}$
  Row index: i_3; Column index: $(i_1-1)I_2+i_2$

$$\Theta_{(3)} \in \mathbb{C}^{3\times(2\times 2)};\ \Theta_{(3)} = \left(\begin{array}{cc|cc} \vartheta_{111} & \vartheta_{121} & \vartheta_{211} & \vartheta_{221} \\ \vartheta_{112} & \vartheta_{122} & \vartheta_{212} & \vartheta_{222} \\ \vartheta_{113} & \vartheta_{123} & \vartheta_{213} & \vartheta_{223} \end{array}\right)$$

2.3 Schmidt decomposition and HOSVD

Now we take a step back and try to focus on two-qubit systems first. Since the tensor elements for two qubits are basically matrix elements (10), we can introduce matrix operations on it. While eigenvalues exist only for square matrices, a more general approach is to consider Schmidt decomposition, where the existence of Schmidt coefficients is guaranteed due to the existence of singular value decomposition (SVD).

Theorem 1 (Schmidt decomposition)**.** Suppose $|\psi_{AB}\rangle$ is the pure state of a composite quantum system of two finite dimensional subsystems A and B. Let M_1 and M_2 be the dimension of the subsystems A and B respectively. Then, there exist orthonormal states $|i_A\rangle$ for subsystem A, $|i_B\rangle$ for subsystem B such that

$$|\psi_{AB}\rangle = \sum_{i=1}^{M_1 M_2} \lambda_i \, |i_A i_B\rangle \, , \tag{13}$$

where λ_i are non-negative real numbers satisfying $\sum\limits_{i=1}^{M_1 M_2} \lambda_i^2 = 1$ and are known as Schmidt coefficients (Nielsen and Chuang, 2000).

Let Λ_{AB} be the Schmidt decomposition of a second order tensor Ψ_{AB}. We would like to point out that Schmidt decomposition in equation (13) has the following important properties, i.e.

1. Pseudo-diagonality: Λ_{AB} is a diagonal matrix with Schmidt coefficients at the diagonal entries,

$$\Lambda_{AB} = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_M \end{pmatrix};$$

2. Ordering: Schmidt coefficients are ordered,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M, \tag{14}$$

where $M = min(M_1, M_2)$. While Λ_{AB} is necessarily a diagonal matrix, the ordering property is included as a convention.

In order to generalize Schmidt decomposition, de Lathauwer et al. (2000) relaxed the pseudo-diagonality property into all-orthogonality condition.

Theorem 2 (Higher order singular value decomposition)**.** Let $\mathcal{X}$ be an Nth-order complex tensor, $\mathcal{X} \in \mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes \ldots \otimes \mathbb{C}^{I_N}$. There exists a core tensor $\mathcal{T}$ of $\mathcal{X}$ and a set of unitary matrices $U^{(1)}$, $U^{(2)}$, ..., $U^{(N)}$such that

$$\mathcal{X} = U^{(1)} \otimes U^{(2)} \otimes \ldots \otimes U^{(N)} \mathcal{T}. \tag{15}$$

The core tensor $\mathcal{T}$ is an Nth-order complex tensor of which the subtensors $\mathcal{T}_{i_n=\alpha}$, obtained by fixing the n-th index to α, have the properties of

1. All-orthogonality: Two subtensors $\mathcal{T}_{i_n=\alpha}$ and $\mathcal{T}_{i_n=\beta}$ are orthogonal for all possible values of n, α and β subject to $\alpha \neq \beta$:

$$\langle \mathcal{T}_{i_n=\alpha}, \mathcal{T}_{i_n=\beta} \rangle = 0 \text{ when } \alpha \neq \beta; \tag{16}$$

2. Ordering:

$$|\mathcal{T}_{i_n=1}| \geq |\mathcal{T}_{i_n=2}| \geq \ldots \geq |\mathcal{T}_{i_n=I_n}| \geq 0 \tag{17}$$

   for all possible values of n,

where $|\mathcal{T}_{i_n=i}| = \sqrt{\langle \mathcal{T}_i, \mathcal{T}_i \rangle}$ is the Frobenius-norm and is called the n-mode singular value of $\mathcal{X}$, $\sigma_i^{(n)}$. The vector $u_i^{(n)}$ is an i-th n-mode singular vector for the respective n-mode singular value of $\mathcal{X}$ (de Lathauwer et al. 2000; Jun-Li and Cong-Feng 2013).

The all-orthogonality condition and n-mode singular value of $\mathcal{X}$, $\sigma_i^{(n)}$ are often combined together (Bin et al., 2012),

$$\langle \mathcal{T}_{i_n=\alpha}, \mathcal{T}_{i_n=\beta} \rangle = \delta_{ij} (\sigma_i^{(n)})^2, \tag{18}$$

where δ_{ij} is the Kronecker's delta.

3. Results and discussions

3.1 HOSVD and two qubits

Although the method of HOSVD is formulated for higher order tensors, it is constructive to see how the relaxed condition on pseudo-diagonality (13) into all-orthogonality (16) will affect the outcome of LU classification for two qubits as the similar effects may appear in the LU classification of multipartite quantum states.

Following the rules of matrix unfolding, Ψ can be written as

- First matrix unfolding, $\Psi_{(1)}$
  Row index: i_1, Column index: i_2

$$\Psi_{(1)} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} \tag{19}$$

- Second matrix unfolding, $\Psi_{(2)}$
  Row index: i_2, Column index: i_1

$$\Psi_{(2)} = \begin{pmatrix} \psi_{11} & \psi_{21} \\ \psi_{12} & \psi_{22} \end{pmatrix} \tag{20}$$

It can be seen that in two-qubit case, matrix unfolding for higher order tensors reduces to the usual matrix transpose operation. This is anticipated as the matrix unfoldings $\Psi_{(1)}$ and $\Psi_{(2)}$ are elements of the space $\mathbb{C}^{2\times(2)}$.

Let ρ^{AB} be the density matrix of the two-qubit pure states, formed by

$$\rho^{AB} = |\psi\rangle \langle\psi| \, . \tag{21}$$

One-particle reduced density matrices, ρ^A and ρ^B, can be obtained by partial trace operation of ρ^{AB},

$$\rho^A = Tr_B(\rho^{AB}), \tag{22}$$
$$\rho^B = Tr_A(\rho^{AB}). \tag{23}$$

It is found that the matrix unfoldings $\Psi_{(1)}$ and $\Psi_{(2)}$ are related to ρ^A and ρ^B through the following relationships:

$$\begin{aligned}\rho^A &= \begin{pmatrix} |\psi_{11}|^2 + |\psi_{12}|^2 & \psi_{11}\bar{\psi}_{21} + \psi_{12}\bar{\psi}_{22} \\ \psi_{21}\bar{\psi}_{11} + \psi_{22}\bar{\psi}_{12} & |\psi_{21}|^2 + |\psi_{22}|^2 \end{pmatrix} \\ &= \Psi_{(2)}^T \bar{\Psi}_{(2)} \\ &= \Psi_{(1)} \Psi_{(1)}^\dagger, \end{aligned} \tag{24}$$

$$\begin{aligned}\rho^B &= \begin{pmatrix} |\psi_{11}|^2 + |\psi_{21}|^2 & \psi_{11}\bar{\psi}_{12} + \psi_{21}\bar{\psi}_{22} \\ \psi_{12}\bar{\psi}_{11} + \psi_{22}\bar{\psi}_{21} & |\psi_{12}|^2 + |\psi_{22}|^2 \end{pmatrix} \\ &= \Psi_{(1)}^T \bar{\Psi}_{(1)} \\ &= \Psi_{(2)} \Psi_{(2)}^\dagger. \end{aligned} \tag{25}$$

From equation (15), the HOSVD of Ψ is written as

$$\Psi = U^{(1)} \otimes U^{(2)} \mathcal{T}, \tag{26}$$

where $\mathcal{T}$ is now representing the core tensor of Ψ. Let $T_{(1)}$ and $T_{(2)}$ be the matrix unfoldings of the core tensor $\mathcal{T}$. The HOSVD of Ψ then reduces to

$$\Psi_{(1)} = U^{(1)} T_{(1)} U^{(2)T}, \tag{27}$$
$$\Psi_{(2)} = U^{(2)} T_{(2)} U^{(1)T}. \tag{28}$$

The all-orthogonality conditions are given as

$$\bar{t}_{11} t_{12} + \bar{t}_{21} t_{22} = 0, \tag{29}$$
$$\bar{t}_{11} t_{21} + \bar{t}_{12} t_{22} = 0, \tag{30}$$

where t_{11}, t_{12}, t_{21}, $t_{22} \in \mathcal{T}$.

By definition, the n-mode singular values of Ψ are

$$\sigma_1^{(1)} = \sqrt{|t_{11}|^2 + |t_{12}|^2}, \tag{31}$$

$$\sigma_2^{(1)} = \sqrt{|t_{21}|^2 + |t_{22}|^2}; \tag{32}$$

$$\sigma_1^{(2)} = \sqrt{|t_{11}|^2 + |t_{21}|^2}, \tag{33}$$

$$\sigma_2^{(2)} = \sqrt{|t_{12}|^2 + |t_{22}|^2}. \tag{34}$$

Note that the squared sum of the singular values for a particular matrix unfolding should be equal to 1 as a consequence of equation (6), i.e.

$$\sigma_1^{(1)2} + \sigma_2^{(1)2} = 1, \tag{35}$$

$$\sigma_1^{(2)2} + \sigma_2^{(2)2} = 1. \tag{36}$$

From equations (24), (25), (27) to (34), we can see that

$$\rho^A = U^{(1)} T_{(1)} T_{(1)}^{\dagger} U^{(1)\dagger} = U^{(1)} \rho_d^A U^{(1)\dagger}, \tag{37}$$

$$\rho^B = U^{(2)} T_{(2)} T_{(2)}^{\dagger} U^{(2)\dagger} = U^{(2)} \rho_d^B U^{(2)\dagger}, \tag{38}$$

where $\rho_d^A = T_{(1)} T_{(1)}^{\dagger}$ and $\rho_d^B = T_{(2)} T_{(2)}^{\dagger}$ are the diagonalized one-particle reduced density matrix for the respective subsystems A and B. In fact, 1-mode and 2-mode singular values correspond to the eigenvalues of ρ^A and ρ^B respectively, while $U^{(1)}$ and $U^{(2)}$ diagonalizes ρ^A and ρ^B respectively (Lipschutz and Lipson, 2008).

3.2 LU classifications

The separable states for two qubits are given as the product states of two individual qubits. Let

$$|\psi_A\rangle = a\,|1\rangle + b\,|2\rangle\,, \tag{39}$$

$$|\psi_B\rangle = c\,|1\rangle + d\,|2\rangle\,, \tag{40}$$

then the separable states for two qubits can be written as

$$\begin{aligned} |\psi_{sep}\rangle &= |\psi_A\rangle \otimes |\psi_B\rangle \\ &= (a\,|1\rangle + b\,|2\rangle) \otimes (c\,|1\rangle + d\,|2\rangle) \\ &= ac\,|11\rangle + ad\,|12\rangle + bc\,|21\rangle + bd\,|22\rangle\,. \end{aligned} \tag{41}$$

For separable states, the all-orthogonality conditions (29) and (30) will become

$$\bar{a}b(|c|^2 + |d|^2) = 0, \tag{42}$$
$$\bar{c}d(|a|^2 + |b|^2) = 0. \tag{43}$$

a and b cannot be both simultaneously zero or else $|\psi_A\rangle$ does not exist. Similar argument applies to c and d. Therefore, if we choose $b = d = 0$ by convention,

$$\sigma_1^{(1)} = \sqrt{|ac|^2} = 1, \tag{44}$$
$$\sigma_2^{(1)} = 0, \tag{45}$$
$$\sigma_1^{(2)} = \sqrt{|ac|^2} = 1, \tag{46}$$
$$\sigma_2^{(2)} = 0. \tag{47}$$

Thus for separable states, the n-mode singular values are 1 and 0.

The above demonstration showed that the method of HOSVD can identify between separable and entangled states of two qubits. However, since HOSVD does not provide a canonical form, we cannot follow a similar approach used by Carteret and Sudbery (2000). To proceed, one sensible consideration is to preserve the algebraic structures of HOSVD after the LU group action. Equation (18) captured such important algebraic structures of HOSVD.

Consider the action of $(U, V) \in SU(2) \times SU(2)$ on the two-qubit states. We do not use the LU group of $U(1) \times SU(2) \times SU(2)$ because complex conjugation operation in equation (18) will cancel off the phase factor of $U(1)$. The action of $(U, V) \in SU(2) \times SU(2)$ on the two-qubit states is

$$\begin{aligned}
U \otimes V |\psi\rangle &= \sum_{ij=1}^{2} \sum_{kl=1}^{2} \sum_{mn=1}^{2} \psi_{ij} |kl\rangle \langle kl| U \otimes V |mn\rangle \langle mn|ij\rangle \\
&= \sum_{ij=1}^{2} \sum_{kl=1}^{2} \psi_{ij} |kl\rangle \langle kl| U \otimes V |ij\rangle \\
&= \sum_{ij=1}^{2} \sum_{kl=1}^{2} u_{ki}\psi_{ij}v_{lj} |kl\rangle = \sum_{kl=1}^{2} \sum_{ij=1}^{2} u_{ki}\psi_{ij}v^T_{jl} |kl\rangle \\
&= \sum_{kl=1}^{2} \psi'_{kl} |kl\rangle ,
\end{aligned}$$

where $\psi'_{kl} = \sum_{ij=1}^{2} u_{ki}\psi_{ij}v^T_{jl}$. With respect to $T_{(1)}$, equation above can then be rephrased as

$$T'_{(1)} = UT_{(1)}V^T = \begin{pmatrix} u_{11} & u_{12} \\ -\bar{u}_{12} & \bar{u}_{11} \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} v_{11} & -\bar{v}_{12} \\ v_{12} & \bar{v}_{11} \end{pmatrix}$$
$$= \begin{pmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{pmatrix}, \tag{48}$$

where the primed elements denote the elements after the group action.

Since the group action on the two-qubit states must preserve equation (18), therefore the all-orthogonality conditions become,

$$\bar{t}'_{11}t'_{21} + \bar{t}'_{12}t'_{22} = 0 \Rightarrow \bar{u}_{11}\bar{u}_{12}(\sigma_1^{(1)} - \sigma_2^{(1)}) = 0. \tag{49}$$
$$\bar{t}'_{11}t'_{12} + \bar{t}'_{21}t'_{22} = 0 \Rightarrow \bar{v}_{11}\bar{v}_{12}(\sigma_1^{(2)} - \sigma_2^{(2)}) = 0, \tag{50}$$

Similarly with the n-mode singular values,

$$\sigma_1^{'(1)2} = |t'_{11}|^2 + |t'_{12}|^2 = |u_{11}|^2 \sigma_1^{(1)2} + |u_{12}|^2 \sigma_2^{(1)2} = \sigma_1^{(1)2}, \tag{51}$$
$$\sigma_2^{'(1)2} = |t'_{21}|^2 + |t'_{22}|^2 = |u_{11}|^2 \sigma_2^{(1)2} + |u_{12}|^2 \sigma_1^{(1)2} = \sigma_2^{(1)2}, \tag{52}$$
$$\sigma_1^{'(2)2} = |t'_{11}|^2 + |t'_{21}|^2 = |v_{11}|^2 \sigma_1^{(2)2} + |v_{12}|^2 \sigma_2^{(2)2} = \sigma_1^{(2)2}, \tag{53}$$
$$\sigma_2^{'(2)2} = |t'_{12}|^2 + |t'_{22}|^2 = |v_{11}|^2 \sigma_2^{(2)2} + |v_{12}|^2 \sigma_1^{(2)2} = \sigma_2^{(2)2}. \tag{54}$$

From equations (49), (51) and (52), we can see that if the 1-mode singular values are not the same, i.e. $\sigma_1^{(1)2} \neq \sigma_2^{(1)2}$, then $u_{12} = 0$. For the second matrix unfolding, equations (50), (53) and (54) showed that if the 2-mode singular values are not equal ($\sigma_1^{(2)2} \neq \sigma_2^{(2)2}$), $v_{12} = 0$. On the other hand, if the 1-mode singular values are equal ($\sigma_1^{(1)2} = \sigma_2^{(1)2}$), we do not have restrictions on U. Similar argument applies to the second matrix unfolding. The stabilizer group for each of the entanglement classes is summarized below:

1. General, ($\sigma_1^{(1)2} > \sigma_2^{(1)2}$, $\sigma_1^{(2)2} > \sigma_2^{(2)2}$):

   $$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, V = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{55}$$

2. Unentangled, ($\sigma_1^{(1)2} = 1$, $\sigma_1^{(2)2} = 1$):

   $$U = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, V = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{56}$$

3. Maximally entangled, $(\sigma_1^{(1)2} = \sigma_2^{(1)2}, \sigma_1^{(2)2} = \sigma_2^{(2)2})$:

$$U = \begin{pmatrix} u_{11} & u_{12} \\ -\bar{u_{12}} & \bar{u_{11}} \end{pmatrix},\ V = \begin{pmatrix} v_{11} & v_{12} \\ -\bar{v_{12}} & \bar{v_{11}} \end{pmatrix} \tag{57}$$

For bipartite quantum systems, its one-particle reduced density matrices are iso-spectral (Klyachko, 2006). Therefore, our list has exhausted all the possible combinations of $\sigma_i^{(n)2}$s.

To compare, we provide Carteret and Sudbery's LU classification results on two qubits below:

1. General, $(\lambda_1 \neq \lambda_2)$:

$$\theta = 0,\ U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix},\ V = \bar{U} \tag{58}$$

2. Unentangled, $(\lambda_1 = 1,\ \lambda_2 = 0)$:

$$e^{i\theta},\ U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix},\ V = \begin{pmatrix} e^{-i(\phi+\theta)} & 0 \\ 0 & e^{i(\phi+\theta)} \end{pmatrix} \tag{59}$$

3. Maximally entangled, $(\lambda_1 = \lambda_2)$:

$$\theta = n\pi,\ U = \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix},\ V = \pm\bar{U} \tag{60}$$

In the LU classification of two qubits by Carteret and Sudbery (2000), the phase factor $U(1)$ differentiates the stabilizer groups of general and unentangled entanglement classes. Since the LU group action used in this work is $SU(2) \times SU(2)$, the general and unentangled entanglement classes can only be differentiated through their first- and second-mode singular values. Also, note that the elements $U \in SU(2)$ and $V \in SU(2)$ are complex-conjugate related in Carteret and Sudbery's work, but this relationship is destroyed when Schmidt decomposition is substituted by HOSVD. Despite that the stabilizer group for each of the entanglement classes is not unique in our case, the distribution of 1-mode and 2-mode singular values between the three entanglement classes are distinct and can be used to differentiate between the three entanglement classes. In this sense, our work agrees with the results anticipated in the paper by Jun-Li and Cong-Feng (2013).

We conclude that by having relaxation on pseudo-diagonality in Schmidt decomposition, we lose information on the complex conjugate relationship between the group elements $U, V \in SU(2)$ which acts on the first and second qubit respectively. Meanwhile, the complex conjugate operation in (18) renders the same stabilizer group for the general and unentangled entanglement classes. Further studies need to be done in order to refine our LU classification results on two qubits to match with the results posted by Carteret and Sudbery.

4. Conclusion

The two-qubit states decomposed by higher order singular value decomposition (HOSVD) can be used to distinguish between the three entanglement classes by comparing between the n-mode singular values. When the local unitary (LU) orbits are calculated, we found that due to the relaxation of pseudo-diagonality of Schmidt decomposition and complex conjugation operation in the algebraic structures of HOSVD, the stabilizer group is not unique for each of the entanglement classes.

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